

Finite-size behaviour of generalized susceptibilities in the whole phase plane of the Potts model

Xue Pan,^{1,2,*} Yanhua Zhang,² Lizhu Chen,³ Mingmei Xu,² and Yuanfang Wu^{2,†}

¹*School of Electronic Engineering, Chengdu Technological University, Chengdu 611730, China*

²*Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan 430079, China*

³*School of Physics and Optoelectronic Engineering, Nanjing University of Information Science and Technology, Nanjing 210044, China*

Using the three-dimensional three-state Potts model, which has the same $Z(3)$ global symmetry as that of the QCD system, we study the sign distribution of generalized susceptibilities in the whole phase plane, and the fluctuations of generalized susceptibilities nearby the phase transition line. The sign change and non-monotonic fluctuations are observable in a small area nearby the phase transition line. A bit further away from the phase transition line, the sign of odd-order susceptibility is opposite in the symmetry (disorder) and broken (order) phases, and that of the even-order one is the same positive in the two phases. So the sign of odd-order susceptibility can be served as a probe of phase transition. It is also demonstrated that the phase boundary of Potts model can be identified by the double logarithm plot of the generalized susceptibilities versus the system size.

PACS numbers: 25.75.Nq; 05.50.+q; 64.60.-i; 24.60.-k;

I. INTRODUCTION

One of the main goals of heavy ion experiments is to locate the critical point, and/or the boundary of Quantum Chromodynamics (QCD) phase transition [1]. An expected QCD phase diagram in the temperature and baryon chemical potential plane is sketched in Fig. 1(a). Where the system undergoes a first-order phase transition at high baryon chemical potential (μ_B) and low temperature [2–4]. With the decrease of μ_B and increase of temperature, the first-order phase transition line ends at a critical point, which belongs to the three-dimensional Ising universality class [5, 6]. At high temperature and vanishing chemical potential, the calculations of lattice QCD have shown it is a crossover [7].

To probe this expected phase diagram in heavy ion experiments, the high-order cumulants of the conserved charges are suggested as sensitive observables of the critical fluctuations [5, 8–12]. Theoretically, they are corresponding to the generalized susceptibilities, and can be calculated by lattice QCD at vanishing μ_B [10, 13–15], and QCD effective models at finite μ_B [16–18]. Both of them show non-monotonic fluctuations

in the vicinity of the critical point, i.e., peak-like structure of the second-order susceptibility, sign change and oscillation for high-order susceptibilities. So non-monotonic behavior of high-order cumulants of conserved charges is considered as a signal of the critical point [19–22].

However, using the three-dimensional three-state Potts model, it has been demonstrated that those critical related fluctuations also appear at other points of the first-order phase transition line and crossover [23]. They are not specified to the critical point, but observable at all points on the phase transition line and crossover region [23, 24]. The order of the phase transition has to be determined by the value of the exponent of finite-size scaling of generalized susceptibilities [23, 25–27].

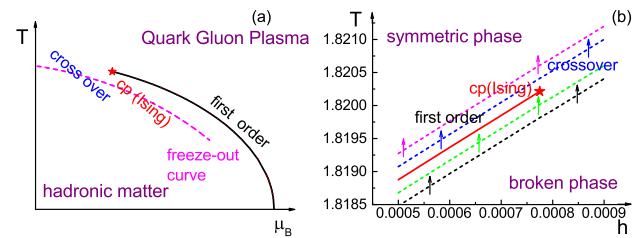


FIG. 1: (Color online). The cartoon of phase diagrams of QCD on the temperature-baryon chemical potential plane (a), and the three-dimensional three-state Potts model on the temperature-external magnetic field plane.

*Electronic address: panxue1624@163.com

†Electronic address: wuyf@mail.ccnu.edu.cn

Now if we go beyond the phase transition

line, new questions are raised. Do these non-monotonic fluctuations still exist nearby the line? What would be left if a bit further? If they survive nearby the transition line, how the fluctuations look like along the lines which are nearby and parallel to the transition line, such as the freeze-out line as showed by violet dashed line in Fig. 1(a), and violet, blue, green and black dashed lines in Fig. 1(b)? Then, how large the area is, and how to identify the phase transition line in the area? Last, but not least, if some signatures remain a bit further away from the line, do they associate with different phases, or phase transition?

So in this paper, using the three-dimensional three-state Potts model, we will show sign distributions of high-order susceptibilities in the whole phase plane, and the fluctuations of the second-order and fourth-order susceptibilities nearby the phase transition line.

The paper is organized as follows. In Section II, the formalism of the generalized susceptibilities in the framework of the Potts model is described and derived.

In Section III, the sign distribution of the high-order susceptibilities in the whole phase plane is firstly presented and discussed. A bit further away from the phase transition line, the signs of odd-order susceptibilities at two sides of the transition line keep the opposite, while those of the even-order ones are the same. The odd-order susceptibility is therefore recommended as a probe of the phase transition. Then, the fluctuations nearby the phase transition line are presented. It is showed that the sign change and non-monotonic fluctuations are still observable in a small area nearby the phase transition line.

In order to identify the phase transition line, double logarithm plots of generalized susceptibilities versus the system size nearby the phase transition temperatures are suggested in Section IV. It is showed that only the plot at transition temperature is a straight line, and others are not. So the double logarithm plot can be served as an identification of the phase transition line. Finally, the summary and conclusions are given in Section V.

II. GENERALIZED SUSCEPTIBILITIES IN THE FRAMEWORK OF THE POTTS MODEL

The partition function of the Potts model is defined as [28],

$$Z(\beta, h) = \sum_{\{s_i\}} e^{-(\beta E - h M)}, \quad (1)$$

where $s_i \in \{1, 2, 3\}$ is the spin at site i of a three-dimensional cubic lattice. $\beta = 1/T$ is the reciprocal of temperature, and $h = \beta H$ is the normalized external magnetic field. E and M denote the energy and magnetization respectively, i.e.,

$$E = -J \sum_{\langle i, j \rangle} \delta(s_i, s_j), M = \sum_i \delta(s_i, s_g). \quad (2)$$

J is an interaction energy between nearest-neighbour spins $\langle i, j \rangle$, and is set up as 1 in our calculations. s_g is the direction of the ghost spin, where the magnetization of non-vanishing external field $h > 0$ prefers to. The order parameter of the system is defined as [29]

$$m = \frac{3}{2} \frac{\langle M \rangle}{V} - \frac{1}{2}, \quad (3)$$

where $V = L^3$ and L is the number of lattice points in each direction. $\langle M \rangle$ is the mean of the whole generated samples.

At vanishing external magnetic field $h = 0$, the model is expected to undergo a temperature driven first-order phase transition due to the spontaneous symmetry breaking. It is similar to the pure gauge QCD theory, where the corresponding order parameter is the Polyakov loop. It plays the role of magnetization in the spin models, signaling the spontaneous breaking of center symmetry of the deconfined phase.

The susceptibility χ_2 constructed from the magnetization M can be obtained from the second-order derivative of the (reduced) free energy density ($f = -\frac{1}{V} \ln Z$) with respect to the external magnetic field,

$$\chi_2 = -\left. \frac{\partial^2 f}{\partial h^2} \right|_T = \frac{1}{V} (\langle M^2 \rangle - \langle M \rangle^2). \quad (4)$$

Without $1/V$, the right part of Eq. (4) is the second-order cumulant of the magnetization.

The high-order derivatives of the free energy density with respect to h are the corresponding generalized susceptibilities of the magnetization. The n th-order susceptibility is as follows,

$$\chi_n = -\left.\frac{\partial^n f}{\partial h^n}\right|_T. \quad (5)$$

For $n = 3, 4, 5$, and 6 , we have,

$$\chi_3 = \frac{1}{V}\langle\delta M^3\rangle, \quad (6)$$

$$\chi_4 = \frac{1}{V}(\langle\delta M^4\rangle - 3\langle\delta M^2\rangle^2), \quad (7)$$

$$\chi_5 = \frac{1}{V}(\langle\delta M^5\rangle - 10\langle\delta M^3\rangle\langle\delta M^2\rangle), \quad (8)$$

$$\chi_6 = \frac{1}{V}(\langle\delta M^6\rangle - 10\langle\delta M^3\rangle^2 + 30\langle\delta M^2\rangle^3 - 15\langle\delta M^4\rangle\langle\delta M^2\rangle), \quad (9)$$

where $\delta M = M - \langle M \rangle$.

In this paper, the estimation of the generalized susceptibilities are based on the Monte Carlo simulation of the three-dimensional three-state Potts model, which is performed on the lattice sizes of L^3 by the Wolff cluster algorithm [30]. The helical boundary conditions are used. For each pair of couplings ($\beta = 1/T, h$), we produce total 50000 independent configurations.

III. FLUCTUATIONS OF GENERALIZED SUSCEPTIBILITIES IN THE WHOLE PHASE PLANE

The phase diagram of the three-dimensional three-state Potts model in the phase plane of temperature and external field is presented in Fig. 1(b). At vanishing external field, the temperature-driven phase transition in the Potts model has proven to be the first-order [31, 32], as showed by solid red line in Fig. 1(b) and called as the phase transition line. It separates the whole phase plane into broken (ordered) and symmetric (disordered) phases. With increase of the external field, the first-order phase transition weakens and ends at a critical point (β_c, h_c)

$= 0.54938(2), 0.000775(10)$), as showed by red star in Fig. 1(b), which belongs to the three-dimensional Ising universality class [28, 33], the same as that of QCD system. Beyond the critical point, it is a crossover.

Fluctuations of generalized susceptibilities are sensitive to the location of the phase transition line. As it has been shown in Fig. 2 of Ref. [23], at a given external field and system size, χ_2 has a peak with the change of temperature. The position of the peak indicates the pseudo phase transition temperature. With the increase of system size, the pseudo phase transition temperature moves towards the phase transition one, which corresponds to the infinite system size. When system size is large enough, such as $L = 60$, the temperature at the peak position is already very close to the phase transition one. Therefore, at a given external magnetic field, the phase transition temperature can be approximately determined by the position of the peak of χ_2 with a large enough system size. The solid red line in Fig. 1(b) indicates the phase transition line, where the external field extends from $h = 0.0003$ to $h = 0.000775$, and the system size $L=60$.

It is clear that in Fig. 2 of Ref. [23], when the temperature is lower (higher) than the phase transition one, χ_2 increases (decreases) monotonously with temperature at a given external field. Moreover, it is always positive. There is no sign change in the whole phase plane.

In contrast, the sign of the third, fourth, fifth, or sixth-order susceptibility, cf. Fig. 3, 4, 5, and 6 of Ref. [23], changes with temperature at a given external field. There is an interval nearby the phase transition temperature that the susceptibilities are negative. So their sign distributions in the whole phase plane are interested and helpful in locating the phase boundary.

A. Sign distribution in the whole phase plane

The sign distributions of $\chi_{3,4,5,6}$ on the whole $T-h$ plane for system size $L = 60$ are presented in Fig. 2(a), 2(b), 2(c), and 2(d), respectively. Where the green and yellow areas correspond to the positive and negative values of $\chi_{3,4,5,6}$, re-

spectively. The solid red line presents the first-order phase transition line. It ends at the critical point, the red point. The red dashed line indicates the crossover.

In Fig. 2(a), sign distribution of χ_3 is separated into two parts by the phase transition line. It is positive (green) above the transition line, and negative (yellow) below the transition line. Similarly, in Fig. 2(c), sign distribution of χ_5 is separated into similar two parts by the two narrow bands, one is negative (yellow) above the transition line, and the other one is positive (green) below the transition line. So the sign of χ_5 changes three times. One happens at phase transition line and other two are at the lines nearby the two sides of the phase transition line. A bit further away from the phase transition line, the sign of χ_3 and χ_5 keeps opposite at symmetry and broken phases.

In contrast, as showed in Fig. 2(b) and 2(d), the sign of χ_4 changes twice, while χ_6 four times. The phase transition line is located in a yellow and green band for χ_4 and χ_6 , respectively. a bit further away from the phase transition line, their sign both keeps positive in the two phases.

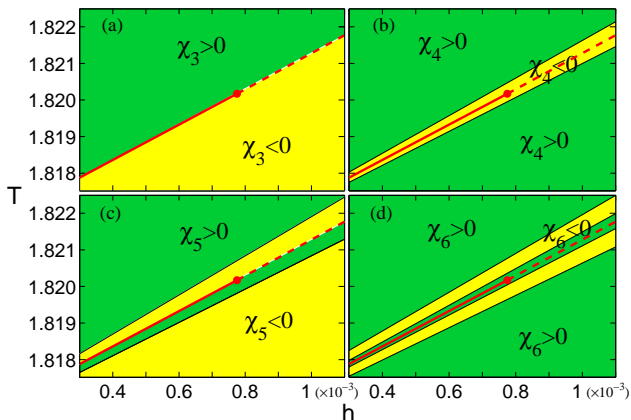


FIG. 2: (Color online). The sign distribution of $\chi_{3,4,5,6}$ on the $T-h$ plane, where the green and yellow areas are positive and negative, respectively. The red solid line presents the phase transition line.

So, in the whole phase space, the sign change happens only nearby the phase transition line. A bit further away from the phase transition line, the sign of odd-order susceptibility keeps positive and negative in the symmetric and broken

phases, respectively. While that of even-order one is the same positive in the two phases. The sign of odd-order susceptibility at high (low) temperature is associated with symmetry (broken) phase in the Potts model. It therefore can be served as an indicator of the phase transition.

This is in fact a common feature of finite-size system as long as there is a phase transition line, such as that showed in Fig. 1(a) and Fig. 1(b) for the QCD system and Potts model, respectively. At each point of phase transition line, the order parameter, i.e., magnetization, as a function of external field is an inflection point (For an infinite-size system, magnetization as a function of external field is discontinuity at first-order phase transition. It turns to be an inflection point in a finite-size system). So its second derivative with respect to the external field, i.e., the third-order susceptibility, is zero and changes the sign once. One more derivative makes the sign change once more [34]. Therefore, the signs of the odd and even-order susceptibilities change odd and even times, respectively.

It is showed in Ref. [34] that the times of sign change of the susceptibility near the critical temperature are determined by the order of the susceptibility. Here, we see this feature keeps not only in the vicinity of the critical point, but all points on the phase transition line.

Turn to the QCD system, if the QCD phase transition exists and the fluctuations of phase transition from the hadronic matter to quark-gluon plasma survive to the final state particles [35], the opposite signs of odd-order cumulants of conserved charges should be expected at the incident energies of SPS and RHIC, which are respectively under and above the phase transition line as estimated [36, 37]. So the measurements of odd-order cumulants of conserved charges at both SPS and RHIC energies are helpful in confirming the phase transition from hadronic matter to quark-gluon plasma.

Certainly, it should also be noticed that if it is far away from the phase transition line, the absolute value of the generalized susceptibilities is very small, cf., Fig. 2 to Fig. 6 of Ref. [23]. In experiments, it may be still difficult to distinguish their signs within experimental errors.

The effective area of the phase transition related fluctuations is relevant to the external field, and the finite system size. The sign distributions of χ_4 , χ_5 , and χ_6 in Fig. 2(b), (c) and (d) show that the width of the bands of sign change with the external field. It becomes wider and wider with increase of external magnetic field. The bigger the external field, the wider the width of the bands.

On the other hand, the larger the system size, the narrower the width of the bands. This feature is presented clearly in Fig. 2 to Fig. 6 of Ref. [23]. For an infinite system, the fluctuation is divergent, and the width of the fluctuation area is zero. The finite system size makes the fluctuation remain in a small temperature region. Then, the questions is how the fluctuations look like nearby the phase transition line?

B. Fluctuations nearby the phase transition line

For the phase transition line (solid red line) of Fig. 1(b), four lines nearby are chosen, i.e., $\beta_{t,h} \pm \Delta\beta$, with $\Delta\beta = 0.00005$, and $\beta_{t,h} \pm 2\Delta\beta$, where $\beta_{t,h} = 1/T_{t,h}$ denotes the reciprocal of the phase transition temperature at the corresponding external magnetic field. On each line, we choose eleven values of h and calculated the corresponding second and fourth-order susceptibilities.

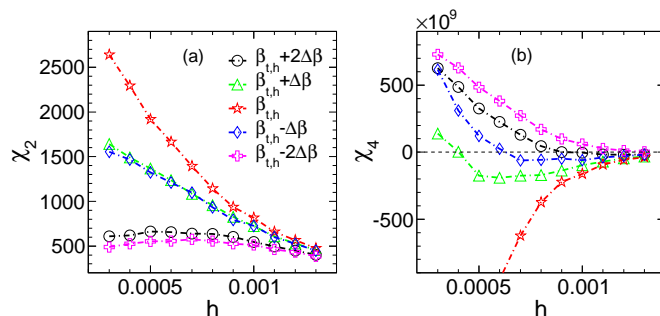


FIG. 3: (Color online). χ_2 (a) and χ_4 (b) on the lines above, along and below the phase transition line, respectively.

χ_2 and χ_4 on these five lines are presented from up to down by black circles, green triangles, red stars, blue diamonds and violet crosses in Fig. 3(a) and 3(b), respectively. It shows that χ_2 in Fig. 3(a) on the phase transition

line (red stars) decreases monotonously with the external field. So do green triangles and blue diamonds with the temperatures deviate $\Delta\beta$ from the phase transition one at each magnetic field. When temperatures deviate $2\Delta\beta$ from the phase transition one, i.e., black circles and violet crosses, there seems a small peak. So even if on the phase transition line, or very close and parallel to the transition line, it is hard to see the non-monotonic fluctuations of the second-order susceptibility.

χ_4 on the transition line, the red stars as showed in Fig. 3(b), it is negative for all of the given external magnetic fields and increases monotonously along the phase transition line. When temperature deviates $\Delta\beta$ from the phase transition one, χ_4 changes from positive to negative at the first-order phase transition region as showed by green triangles and blue diamond. There is a negative valley. When it deviates $2\Delta\beta$ from the phase transition temperature, as black circles and violet crosses showed in Fig. 3(b), the values of χ_4 are all positive. Therefore, when it is not far away from the phase transition line, the sign change and non-monotonic fluctuations of χ_4 are still observable even if it does not pass through the phase transition line.

The left question is how to identify the phase transition line from the lines which are nearby it. The finite-size behaviour of the susceptibilities should be helpful to answer this question [23, 25, 26].

IV. IDENTIFICATION OF THE PHASE TRANSITION LINE

The finite-size scaling law for second-order phase transitions is well built up in statistical physics [38, 39]. Similar finite-size scaling behavior is also found in the first-order phase transitions [40]. The difference is the value of the scaling exponent for each thermodynamic quantity. In the crossover region, the thermodynamic quantities are system size independent. These characteristics have been used in identifying the crossover at vanishing μ_B in lattice QCD calculations [7], and the critical point of nuclear fragmentation [41, 42].

In the vicinity of the critical point, χ_2 is a function of temperature and system size. It follows the finite-size scaling relation [43]

$$\chi_2(T, L) = L^{\gamma/\nu} F_{\chi_2}(\tau L^{1/\nu}). \quad (10)$$

Where $\tau = \frac{T-T_c}{T_c}$ is the reduced temperature. γ and ν are the critical exponents of χ_2 and the correlation length, respectively. F_{χ_2} is the scaling function with scaled variable $\tau L^{1/\nu}$. At the critical point $\tau = 0$, the scaling function F_{χ_2} is a constant independent of L , i.e.,

$$\chi_2(L) \propto L^a, \quad (11)$$

where $a = \gamma/\nu$ is the scaling exponent.

For a first-order phase transition, the relation in Eq. (11) is also applicative at the phase transition temperature. Generally, $a = \gamma/\nu$ is a fraction between zero and the space dimension d of the system for a second-order phase transition. It is the space dimension d for a first-order phase transition. When a first-order phase transition is very weak, it will be smaller than d , and shows the character of a second-order phase transition [44]. This is just the case of the three-dimensional three-state Potts model as showed in Ref. [23]. For a crossover, it is zero [7].

If we take the logarithm of Eq. (11), then

$$\ln \chi_2(L) = a \ln L + C_1, \quad (12)$$

where C_1 is a constant. So if along the phase transition line, the double logarithm plot of χ_2 versus L is a straight line. When the temperature deviates from the phase transition one, the scaling function is system size dependent. $\ln \chi_2$ is no longer a linear function of $\ln L$.

In Fig. 4(a), for an arbitrary external field $h = 0.0005$ in the region of first-order phase transition, $\ln \chi_2$ versus $\ln L$ is presented at phase transition temperature $\beta_t = 0.54979$, and two nearby temperatures, i.e., $\beta_t - \Delta\beta$ with $\Delta\beta = 0.00005$ and $\beta_t - 2\Delta\beta$ by the red stars, blue squares and black points, respectively. Where five system sizes are $L = 40, 45, 50, 55$, and 60 . It is clearly showed that only the red connecting line is a straight line. The blue and black lines are both downward bending, and the black one bends more than that of the blue one, i.e., the farther

the temperature away from the transition one, the more the corresponding line bends. Therefore, for a fixed external field, whether the double logarithm plot of χ_2 versus L at a given temperature is a straight line or not can be served as a judgement of the phase transition temperature.

For generalized susceptibilities, analogous finite-size scaling is expected [23]. The difference is the slope of the straight line. The higher the order of the susceptibility, the bigger the slope of the line is. In Fig. 4(b), we present the logarithm of the absolute value of χ_4 , $\ln(|\chi_4|)$ as a function of $\ln L$ at the same external field $h = 0.0005$, the phase transition temperature β_t , and two more nearby temperatures, $\beta_t - 0.5\Delta\beta$ and $\beta_t - \Delta\beta$. They are showed by the red, blue and black lines, respectively. It is demonstrated again that only at the phase transition temperature, the red line is a straight one. The blue and black ones are downward bending. Here, the temperatures of blue and black lines are less away from the transition temperature in comparison to the cases in Fig. 4(a). The deviation from the straight line is obvious although with a big error bar. This shows that the higher the order of the susceptibility, the more sensitive to the change of the temperature.

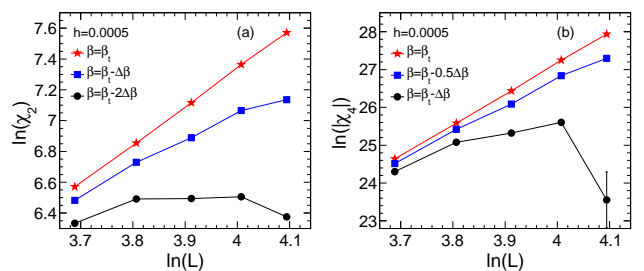


FIG. 4: (Color online). The double logarithm plot of χ_2 (a) and χ_4 (b) as a function of system size L . Where β_t is the reciprocal of phase transition temperature at $h = 0.0005$ and $\Delta\beta = 0.00005$.

V. SUMMARY AND CONCLUSIONS

Using the three-dimensional three-state Potts model, we study generalized susceptibilities of magnetization. Their sign distributions in the whole phase plane and the fluctuations nearby

the phase boundary are presented and discussed.

It is showed that the sign change happens only nearby the phase transition line. A bit further away from the phase transition line, the sign of odd-order susceptibility is positive and negative in the symmetric (disorder) and broken (order) phases, respectively, while the sign of the even-order one keeps positive in the two phases. The opposite sign of odd-order susceptibility can be served as a probe of phase transition. If it can be applied to heavy ion experiments, the measurements of odd-order cumulants of conserved charges at both SPS and RHIC energies would be helpful in confirming the phase transition from hadronic matter to quark-gluon plasma.

It is also showed that the sign change and non-monotonic fluctuations are still observable in a small area nearby the phase transition line. The effective area of the phase transition related fluctuations is dependent on the external field and system size. The bigger the external field and the smaller the system size, the wider the width

of the area is.

It is further demonstrated that only on the phase transition line, the double logarithm plot of the generalized susceptibility versus the system size is a straight line. If the temperature is away from the phase transition one, the plot deviates from a straight line. So the plot is helpful in locating the phase boundary.

VI. ACKNOWLEDGEMENT

This work is supported by the NSFC of China under Grants No. 11647093, 11405088 and 11521064, Fund Project of Sichuan Provincial Department of Education under Grant No. 16ZB0339, Fund Project of Chengdu Technological University under Grant No. 2016RC004, the Major State Basic Research Development Program of China under Grant No. 2014CB845402, and the Ministry of Science and Technology (MoST) under grant No. 2016YFE0104800.

-
- [1] J. Adams *et al.* (STAR Collaboration), Nucl. Phys. A **757**, 102 (2005).
 - [2] P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002).
 - [3] S. Ejiri, Phys. Rev. D **78**, 074507 (2008).
 - [4] E. S. Bowman and J. I. Kapusta, Phys. Rev. C **79**, 015202 (2009).
 - [5] M. Stephanov, K. Rajagopal, E. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998).
 - [6] P. de Forcrand, O. Philipsen, Phys. Rev. Lett. **105**, 152001 (2010).
 - [7] Y. Aoki, G. Endrődi, Z. Fodor, S.D. Katz, K.K. Szabó, Nature **443**, 675 (2006).
 - [8] M. A. Stephanov, K. Rajagopal, and E. V. Shuryak, Phys. Rev. D **60**, 114028 (1999).
 - [9] V. Koch, arXiv:0810.2520.
 - [10] M. Cheng *et al.*, Phys. Rev. D **79**, 074505 (2009).
 - [11] L. Adamczyk *et al.* (STAR Collaboration), Phys. Rev. Lett. **112**, 032302 (2014).
 - [12] L. Adamczyk *et al.* (STAR Collaboration), Phys. Rev. Lett. **113**, 092301 (2014).
 - [13] F. Karsch, Cent. Eur. J. Phys. **10**, 1234 (2012).
 - [14] A. Bazavov *et al.*, Phys. Rev. Lett. **109**, 192302 (2012).
 - [15] R. Bellwied, S. Borsanyi, Z. Fodor, S. D. Katz, A. Pasztor, C. Ratti, K. K. Szabo, Phys. Rev. D **92**, 114505 (2015).
 - [16] M. Asakawa, S. Ejiri, and M. Kitazawa, Phys. Rev. Lett. **103**, 262301 (2009).
 - [17] W.J. Fu and Y.L. Wu, Phys. Rev. D **82**, 074013 (2010).
 - [18] J.W. Chen, J. Deng, H. Kohyama and L. Labun, Phys. Rev. D **93**, 034037 (2016).
 - [19] M. A. Stephanov, Phys. Rev. Lett. **107**, 052301 (2011).
 - [20] B. Friman, F. Karsch, K. Redlich and V. Skokov, Eur. Phys. J. C **71**, 1694 (2011).
 - [21] J.W. Chen, J. Deng, H. Kohyama and L. Labun, Phys. Rev. D **95**, 014038 (2017).
 - [22] X. Pan, L.Z. Chen, X.S. Chen, and Y.F. Wu, Nucl. Phys. A **913**, 206 (2013).
 - [23] X. Pan, M.M. Xu and Y.F. Wu, J. Phys. G: Nucl. Part. Phys. **42**, 015104 (2015).
 - [24] W.k. Fan, X.F. Luo, and H.S. Zong, arXiv:1702.08674.
 - [25] Y.F. Wu, L.Z. Chen, X.S. Chen, *Proc. Sci.*, CPOD2009 (2009) 036; Y.F. Wu, L.Z. Chen, X. Pan, M. Shao, X.S. Chen, Cent. Eur. J. Phys. **10**, 1341 (2012).
 - [26] L. Z. Chen, Y. Y. Chen and Y. F. Wu, Chin. Phys. C **38**, 104103 (2014); X. Pan, L.Z. Chen and Y.F. Wu, Chin. Phys. C **40**, 093104 (2016).
 - [27] Roy A. Lacey, Phys. Rev. Lett. **114**, 142301 (2015); Nucl. Phys. A **956**, 348 (2016); arXiv:1606.08071.
 - [28] A. Patel, Nucl. Phys. B **243**, 411 (1984); A. Patel, Phys. Lett. B **139**, 394 (1984).
 - [29] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
 - [30] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).
 - [31] H. W. J. Blöte, R.H. Swendsen, Phys. Rev. Lett. **43**, 799 (1979).
 - [32] W. Janke, R. Villanova, Nucl. Phys. B **489**, 679 (1997).
 - [33] F. Karsch, S. Stickan, Phys. Lett. B **488**, 319 (2000).
 - [34] M.M. Xu, X. Pan, Y.F. Wu, Nucl. Phys. A **927**, 69 (2014).
 - [35] S. Borsanyi, Z. Fodor, S.D. Katz, S. Krieg, C. Ratti and K.K. Szab, Phys. Rev. Lett. **111**, 062005 (2013); Phys. Rev. Lett. **113**, 052301 (2014) and references therein.

- [36] C. Blume and C. Markert, Prog. Part. Nucl. Phys. **66**, 834-879 (2011); M. Makowiak-Pawowska, Cent. Eur. J. Phys. **10**, 1285 (2012).
- [37] M. M. Aggarwal *et al.* (STAR Collaboration), arXiv:1007.2613.
- [38] V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).
- [39] X.S. Chen, V. Dohm and A.L. Talapov, Physica A: Statistical Mechanics and its Applications **232**, 375 (1996).
- [40] Murty S. S. Challa, D. P. Landau, K. Binder, Phys. Rev. B **34**, 1841 (1986).
- [41] M. Kleine Berkenbusch *et al.*, Phys. Rev. Lett. **88**, 022701 (2001).
- [42] J.B. Elliott *et al.*, Phys. Rev. Lett. **88**, 042701 (2002).
- [43] V. Privman, Finite Size Scaling and Numerical Simulation of Statistical Systems (Singapore: World Scientific) (1990).
- [44] P. Peczak and D. P. Landau Phys. Rev. B **39**, 11932 (1989).